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Uniformly-valid asymptotic solutions to the Orr-Sommerfeld equation using multiple scales

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Abstract. The Orr-Sommerfeld equation is studied using the method of multiple scales. The leading-order uniformly-valid asymptotic solutions are obtained without recourse to heuristic methods for two different types of velocity profiles. Solutions were obtained for arbitrary velocity profiles without inflection or critical points and for profiles with simultaneous inflection and critical points. These are shown to reproduce previously obtained solutions in the inner and outer regions.

1. Introduction

In hydrodynamical stability theory the Orr-Sommerfeld equation

$$\frac{1}{i\alpha R}(\phi^{iv} - 2\alpha^2\phi'' + \alpha^4\phi) = (U - c)(\phi'' - \alpha^2\phi) - U''\phi \tag{1a}$$

with boundary conditions

$$\phi(-1) = \phi'(-1) = \phi(1) = \phi'(1) = 0 \tag{1b}$$

governs two-dimensional, incompressible, parallel flow in a channel, where ϕ = amplitude of disturbance (complex), U = mean velocity (assumed known), α = wave number of disturbance (real), c = wave speed of disturbance (complex), R = Reynolds number of mean flow.

This equation with appropriately modified boundary conditions also governs boundary-layer stability but only locally; since parallel flow is assumed, boundary-layer growth effects are not accounted for.

Due to the nature of the OS equation its solution can be separated into even and odd functions. The odd solutions are found to be stable [1]. We are therefore free to concentrate on the even solutions. These have boundary conditions of the form

$$\phi'(0) = \phi'''(0) = \phi(1) = \phi'(1) = 0, \tag{1c}$$

which are equivalent to those presented above.

Instability is experimentally known to occur at large Reynolds numbers; it is therefore useful to define

$$\varepsilon = \frac{1}{i\alpha R}$$

which can be considered a small quantity. Since this small parameter multiplies the highest derivative, finding the asymptotic solution to (1) is a singular perturbation problem. This also means that trying to solve (1) numerically is a nontrivial problem and can be very costly [2].

Since equations (1a) and (1b) or (1c) constitute an eigenvalue problem, historically it has been assumed sufficient to find approximations to its four linearly independent solutions. Once these are found, the determinantal (or secular) equation for the eigenvalues is formed from the boundary conditions. For channel flow it takes the form:

$$F(\alpha, c, R) = \begin{vmatrix} \phi_1(1) & \phi_2(1) & \phi_3(1) & \phi_4(1) \\ \phi_1'(0) & \phi_2'(0) & \phi_3'(0) & \phi_4'(0) \\ \phi_1'(1) & \phi_2'(1) & \phi_3'(1) & \phi_4'(1) \\ \phi_1'''(0) & \phi_2'''(0) & \phi_3'''(0) & \phi_4'''(0) \end{vmatrix} = 0.$$

For given values of α and R one would attempt to solve this for the complex wave speed c $(c_r + ic_i)$. Typically the curve corresponding to $c_i = 0$ (neutral disturbances) is drawn on a graph with α and R as the axes. This curves defines the border between the stable and unstable solutions. Most importantly, it yields the critical Reynolds number, below which all disturbances are stable.

Asymptotic solutions to the OS equation have been obtained by many researchers. The earliest were by Heisenberg [3], Tollmien [4], and Lin [5]. These and more are reviewed by Lin [6] and Drazin and Reid [1]. The majority of these solutions were obtained somewhat heuristically, thus making them difficult to extend to other problems. The purpose of this paper is to derive asymptotic solutions to the OS equation in a methodical manner. The solutions will be uniformly-valid asymptotic solutions, that is, the governing equations and boundary conditions are satisfied with uniform asymptotic accuracy. This does not imply automatically that they are uniformly valid approximations to the exact solution of the OS equation, although one usually expects this to be true. The solutions will be shown to be equivalent to previous results, but because the method proceeds systematically, it is easily extended to other velocity profiles and even to other problems.

Most previous asymptotic solutions to the OS equation have been of the inner- and outer-expansion type. Apparently only Tam [7] has attempted to use the method of multiple scales. But his results do not appear to be uniformly valid [1, p. 253].

If one formally lets $|\varepsilon| \to 0$ in (1a) it becomes the Rayleigh equation:

$$(U - c)(\phi'' - \alpha^2\phi) - U''\phi = 0.$$

The solutions to this are usually referred to as the "inviscid" or "outer" solutions since they are valid only in the limit of infinite Reynolds number or very far from the walls. While these solutions yield many interesting and important results [6], it is well known that one must be careful not to infer too much about the complete OS equation from the inviscid results.

Analogously, the "viscous" or "inner" equation is obtained by using a stretched variable of the form $\eta = z/\varepsilon$ so that the OS equation becomes

$$\frac{d^4\phi}{d\eta^4} - \varepsilon(U - c)\frac{d^2\phi}{d\eta^2} = 2\alpha^2\varepsilon^2\phi'' - \varepsilon^3[(U - c)\alpha^2\phi + U''\phi] - \alpha^4\varepsilon^4\phi.$$

Neglecting $O(\varepsilon^2)$ terms yields Reid's "truncated" equation [8]:

$$\phi^{iv} - (U - c)\varepsilon\phi'' = 0,$$

or the inner equation if both the independent and dependent variables are transformed using a Langer-type variable.

The difficulty lies in determining four solutions that will approximate the four linearly independent solutions to the actual OS equation. In the past, two asymptotic solutions to the Rayleigh equation were used to approximate two of these. The remaining two approximate solutions were obtained from the inner equation or the WKB method (depending on whether or not critical points exist). These four solutions were used in the secular equation to obtain the neutral curve. The validity of this process is not completely obvious since, strictly speaking, the inner and outer solutions are valid only in the inner and outer regions, respectively.

The current paper will show that by using the method of multiple scales it is possible to obtain approximations to all four linearly independent solutions to the OS equation in one analysis. At the same time their uniform validity will be established. Despite the fact that some of the equations appear unwieldy, the differential equation that is eventually solved is no more difficult to solve than the inner equation. The lengthy algebraic manipulations required for the present study were greatly simplified through the use of the MIT Laboratory for Computer Sciences' MACSYMA system (project MAC's SYmbolic MAnipulation system). Algebraic manipulation programs of this type have been described by Pavelle [9].

2. Velocity profiles with no inflection or critical points

At first it will be assumed that there are no inflection points or critical points in the velocity profile; i.e., U'' and $U - c$ remain $O(1)$ throughout the region of interest. This simplifies the analysis considerably. A more general case which includes these singularities will be discussed later.

In using the multiple-scales method the ordinary differential equation is converted to a partial differential equation by assuming that the solution depends upon two variables instead of one, i.e.,

$$\phi(z) = \phi(z, \eta).$$

All the generality necessary can be obtained by using a stretched variable of the form

$$\eta = \frac{1}{\varepsilon^{\nu}}[g_0(z) + \varepsilon^{|\nu|}g_1(z) + \varepsilon^{|2\nu|}g_2(z)\ldots].$$

However, if one carries out the analysis to be described in this paper, it is straightforward to show that only two scales are necessary to find a uniformly valid zeroth-order solution. That is, $g_1(z)$, $g_2(z)$, etc. must be constant and these constants must be identically zero in order to have [10]

$$\lim_{z\to 0} \eta = \frac{z}{\varepsilon^{\nu}}.$$

Therefore, for the remainder of the paper it will be sufficient to use

$$\eta = \frac{g(z)}{\varepsilon^{\nu}} \tag{2}$$

as the stretched or "fast" variable. Although for velocity profiles other than those discussed in this paper, one may have to re-examine this statement.

The OS equation is converted to a partial differential equation using equation (2) and the chain rule. The ordinary derivatives of ϕ expanded into partial-derivative form become:

$$\frac{d\phi}{dz} = \frac{\partial\phi}{\partial z} + \frac{\partial\phi}{\partial\eta}\frac{d\eta}{dz} = \phi_z + \frac{g'(z)}{\varepsilon^{\nu}}\phi_\eta, \tag{3}$$

$$\frac{d^2\phi}{dz^2} = \phi_{zz} + \frac{g'^2}{\varepsilon^{2\nu}}\phi_{\eta\eta} + \frac{2g'}{\varepsilon^{\nu}}\phi_{\eta z} + \frac{g''\phi_\eta}{\varepsilon^{\nu}}, \tag{4}$$

$$\begin{aligned}\frac{d^4\phi}{dz^4} = {} & \phi_{zzzz} + \frac{g'^4}{\varepsilon^{4\nu}}\phi_{\eta\eta\eta\eta} + \frac{4g'^3}{\varepsilon^{3\nu}}\phi_{\eta\eta\eta z} + \frac{6g'^2g''}{\varepsilon^{3\nu}}\phi_{\eta\eta\eta} + \frac{6g'^2}{e^{2\nu}}\phi_{\eta\eta zz} + \frac{12g'g''}{\varepsilon^{2\nu}}\phi_{\eta\eta z} \\ & + \frac{4g'g'''}{\varepsilon^{2\nu}}\phi_{\eta\eta} + \frac{3g''^2}{\varepsilon^{2\nu}}\phi_{\eta\eta} + \frac{4g'}{\varepsilon^{\nu}}\phi_{\eta zzz} + \frac{6g''}{\varepsilon^{\nu}}\phi_{\eta zz} + \frac{4g'''}{\varepsilon^{\nu}}\phi_{\eta z} + \frac{g^{iv}\phi_\eta}{\varepsilon^{\nu}}.\end{aligned} \tag{5}$$

Replacing the ordinary derivatives in the OS equation with the above relations gives an "expanded" OS equation:

$$\begin{aligned}& \varepsilon^{1-4\nu}g'^4\phi_{\eta\eta\eta\eta} - \varepsilon^{-2\nu}g'^2(U-c)\phi_{\eta\eta} + \varepsilon^{1-3\nu}g'^2[4g'\phi_{\eta\eta\eta z} + 6g''\phi_{\eta\eta\eta}] \\ & - \varepsilon^{-\nu}(U-c)[g''\phi_\eta + 2g'\phi_{\eta z}] - [(U-c)(\phi_{zz} - \alpha^2\phi) - U''\phi] \\ & + \varepsilon^{1-2\nu}[6g'^2\phi_{\eta\eta zz} + 12g'g''\phi_{\eta\eta z} + (4g'g''' + 3g''^2 - 2\alpha^2g'^2)\phi_{\eta\eta}] \\ & + \varepsilon^{1-\nu}[4g'\phi_{\eta zzz} + 6g''\phi_{\eta zz} + 4(g''' - \alpha^2g')\phi_{\eta z} + (g^{iv} - 2\alpha^2g'')\phi_\eta] \\ & + \varepsilon[\phi_{zzzz} - 2\alpha^2\phi_{zz} + \alpha^4\phi] = 0.\end{aligned} \tag{6}$$

The exponent ν is determined at this point. According to Kruskal [11], ν should be chosen so as to "minimize the simplification" of the equation. This means retaining many terms as possible in the zeroth-order governing equation and thus, it is assumed, keeping as much of the physics in the problem as possible. This can be achieved in this case by choosing ν so that the first two terms (the asymptotically largest in the equation) are of the same asymptotic order of magnitude. This yields $\nu = 1/2$, which is consistent with previous results [3].

Substituting $\nu = 1/2$ into (6) gives

$$\varepsilon^{-1}g'^2[g'^2\phi_{\eta\eta\eta\eta} - (U - c)\phi_{\eta\eta}] + \varepsilon^{-1/2}[4g'^3\phi_{\eta\eta\eta z} + 6g'^2g''\phi_{\eta\eta\eta} - (U - c)(2g'\phi_{\eta z} + g''\phi_\eta)]$$
$$+ [-(U - c)\phi_{zz} + 6g'^2\phi_{\eta\eta zz} + 12g'g''\phi_{\eta\eta z} + (\alpha^2(U - c) - U'')\phi$$
$$+ (4g'g''' + 3g''^2 - 2\alpha^2g'^2)\phi_{\eta\eta}] + \varepsilon^{1/2}[4g'\phi_{\eta zzz} + 6g''\phi_{\eta zz} + 4(g''' - \alpha^2g')\phi_{\eta z}$$
$$+ (g^{iv} - 2\alpha^2g'')\phi_\eta] + \varepsilon[\phi_{zzzz} - 2\alpha^2\phi_{zz} + \alpha^4\phi] = 0.$$

Next $\phi(z, \eta)$ is assumed to possess an asymptotic expansion in powers of $\varepsilon^{1/2}$,

$$\phi(z, \eta) = \phi_0(z, \eta) + \varepsilon^{1/2}\phi_1(z, \eta) + \varepsilon\phi_2(z, \eta) + \dots,$$

and this expansion is substituted into (7). The resulting equation is too lengthy to include here; its derivation was greatly facilitated by the use of MIT's MACSYMA system. Since this equation must be valid for all values of ε, the coefficients of all powers of ε must equal zero. This yields the governing equations for the functions $\phi_0, \phi_1, \dots$, in the form:

$$\phi_{0_{\eta\eta\eta\eta}} - K^2\phi_{0_{\eta\eta}} = 0, \tag{8}$$

$$\phi_{1_{\eta\eta\eta\eta}} - K^2\phi_{1_{\eta\eta}} = \frac{1}{g'}\left[-4\phi_{0_{\eta\eta\eta z}} - \frac{6g''}{g'}\phi_{0_{\eta\eta\eta}} + 2K^2\phi_{0_{\eta z}} + \frac{g''}{g'}K^2\phi_{0_\eta}\right], \tag{9}$$

$$\begin{aligned}\phi_{2_{\eta\eta\eta\eta}} - K^2\phi_{2_{\eta\eta}} &= \frac{1}{g'}\left[-4\phi_{1_{\eta\eta\eta z}} - \frac{6g''}{g'}\phi_{1_{\eta\eta\eta}} + 2K^2\phi_{1_{\eta z}} + \frac{g''K^2}{g'}\phi_{1_\eta}\right]\\ &+ \frac{1}{g'^2}\left[-6\phi_{0_{\eta\eta zz}} - \frac{12g''}{g'}\phi_{0_{\eta\eta z}} - \frac{4g'''}{g'}\phi_{0_{\eta\eta}} - \frac{3g''^2}{g'^2}\phi_{0_{\eta\eta}} + 2\alpha^2\phi_{0_{\eta\eta}}\right]\\ &+ \frac{1}{g'^4}[(U - c)(\phi_{0_{zz}} - \alpha^2\phi_0) - U''\phi_0],\end{aligned} \tag{10}$$

where

$$K^2(z) \equiv \frac{U - c}{g'^2}.$$

As is typical of the multiple-scales method, the operator repeats itself at each order and is of the same form as the inner equation. Also, the forcing functions depend solely upon the solutions to the previous-order equations. In theory one should be able to solve the lowest-order problem and then solve the higher-order ones successively. Since the complexity of the forcing functions grows rapidly, this is often impractical. Nevertheless, these are recursion formulae for all higher-order solutions and are of value. It is also important to notice that the Rayleigh operator is contained in the forcing function for ϕ_2 (equation (10)). The significance of this will be demonstrated later on in the analysis. It is therefore readily apparent that both the inner and outer descriptions of the OS equation are contained within the first three orders of the multiple-scale analysis.

Of course the boundary conditions must also be expanded when using a perturbation technique. This would give a set of boundary conditions for each order solution ϕ. It should be noted that only the zeroth-order problem will have homogeneous boundary conditions. This means that only the zeroth-order problem will be an eigenvalue problem – as it should be.

The general solution to (8) is

$$\phi_0(z, \eta) = \frac{A_0(z)}{K(z)^2} e^{K(z)\eta} + \frac{B_0(z)}{K(z)^2} e^{-K(z)\eta} + D_0(z)\eta + E_0(z) \tag{11}$$

where A_0, B_0, D_0, E_0, and K are complex functions.

Conditions which must be placed on the arbitrary coefficients (or functions) in order to insure uniform validity are derived by noting that the expansion

$$\phi = \sum_{n=0}^{\infty} \varepsilon^{n/2} \phi_n$$

is asymptotic only if the ϕ_i's are all of the same order [10], i.e., ϕ_n/ϕ_{n-1} must remain O(1) as $|\eta| \to \infty$. Therefore ϕ_1 must not become any more singular than ϕ_0. It will now be shown how the coefficients $g(z)$, $A_0(z)$, $B_0(z)$, $D_0(z)$, and $E_0(z)$ can be chosen so as to insure uniform validity of ϕ_0 to $O(\varepsilon^{1/2})$.

In order to obtain the necessary boundedness of the solution, it is generally sufficient to eliminate all solutions to the homogeneous form of the equation from the forcing function of the inhomogeneous form of the equation. This will insure that the higher-order solutions do not become any more singular than the zeroth-order solution [12, 13], this can be done at any order if more accurate solutions are desired.

The forcing function in equation (9) is

$$\frac{-4}{g'} \phi_{0_{\eta\eta\eta z}} - \frac{6g''}{g'^2} \phi_{0_{\eta\eta\eta}} + \frac{2(U-c)}{g'^3} \phi_{0_{\eta z}} + \frac{g''(U-c)}{g'^4} \phi_{0_\eta}.$$

Using the solution for ϕ_0 (equation (11)) this becomes

$$\left(6A_0K' + 2A_0'K - 2A_0K'K\eta + \frac{5g''A_0K}{g'}\right) e^{K\eta}$$
$$- \left(6B_0K' + 2B_0'K - 2B_0K'K\eta + \frac{5g''B_0K}{g'}\right) e^{-K\eta} + \left(2D_0' + \frac{g''D_0}{g'}\right) K^2.$$

Since $e^{K\eta}$, $e^{-K\eta}$, and K^2 are all solutions to the homogeneous equation, the entirety of the above expression must be forced to zero; this will insure that ϕ_1/ϕ_0 remains O(1) as $\eta \to \infty$.

Eliminating the above forcing function is accomplished by choosing appropriate values for the arbitrary functions K, A_0, B_0, and D_0. It is easily verified that the following choices force the expression to zero:

$$K' = 0 \quad \text{or} \quad K = \text{constant, say unity,}$$

$$\frac{A_0'}{A_0} + \frac{5}{2}\frac{g''}{g'} = 0, \quad \frac{B_0'}{B_0} + \frac{5}{2}\frac{g''}{g'} = 0, \quad \frac{D_0'}{D_0} + \frac{1}{2}\frac{g''}{g'} = 0.$$

These equations yield

$$g(z) = \int^z \sqrt{U - c}\,\mathrm{d}z \Rightarrow \eta = \int^z \sqrt{i\alpha R(U - c)}\,\mathrm{d}z,$$

$$A_0(z) = a(U - c)^{-5/4}, \quad B_0(z) = b(U - c)^{-5/4}, \quad D_0(z) = d(U - c)^{-1/4},$$

in which a, b, and d are arbitrary complex constants.

The above conditions are necessary for the uniform validity of the solution; but the arbitrary function $E_0(z)$ has not been determined yet. This is unlike many textbook problems in this respect, since most of those involve only second-order differential equations. Since the OS equation is a fourth-order differential equation it is necessary to look at the second-order governing equation in order to determine $E_0(z)$. Specifically the forcing function on ϕ_2 must be investigated (equation (10)). This does not indicate an error in the solution; it simply means that the solution at this point is not valid far enough from the boundary to discover any functional dependence for $E_0(z)$. To the order of accuracy calculated thus far, $E_0(z)$ is essentially unimportant, and one would probably not be able to meet the outer boundary condition with this form of the solution. Therefore it is necessary to investigate the forcing function on ϕ_2 to find $E_0(z)$.

By substituting the solution for ϕ_0 into (10) it is found that in order to insure uniform validity, $E_0(z)$ must satisfy the Rayleigh equation! Since this is the only place $E_0(z)$ occurs in (10) and it is the only term having no functional dependence on η, it must be set equal to zero, as was done earlier for A_0, B_0, and D_0. Thus,

$$(U - c)(E_0'' - \alpha^2 E_0) - U'' E_0 = 0.$$

This is the outer equation as discussed earlier and its presence here should come as no surprise. The important point to make is that it is found in a very methodical manner and allows one to prove uniform validity.

Further, not only does $E_0(z)$ satisfy the Rayleigh equation but $D_0(z)$ must also. But it was found earlier that

$$D_0(z) = d(U - c)^{-1/4}.$$

Therefore in order to extend the validity of the solution out far enough to see the Rayleigh equation and to find $E_0(z)$, it is necessary that $d = 0$ or $D_0(z) = 0$. It is important to note that all four boundary conditions can still be satisfied since $E_0(z)$ satisfies a second-order differential equation.

In summary, assuming $U - c$ and U'' remain O(1), the uniformly-valid leading-order solution to the OS equation is

$$\phi_0(z, \eta) = (U - c)^{-5/4}(ae^{\eta} + be^{-\eta}) + E_{0_1}(z) + E_{0_2}(z) + O(|\varepsilon|^{1/2}) \qquad (12)$$

where

$$(U - c)(E''_{0_{1,2}} - \alpha^2 E_{0_{1,2}}) - U''E_{0_{1,2}} = 0$$

and

$$\eta = \int^z \sqrt{i\alpha R(U - c)}\,dz.$$

This is uniformly valid as $\varepsilon \to 0$ for all z such that $g(z) = O(1)$, which means

$$\frac{\phi_1}{\phi_0} = O(1) \quad \text{and} \quad \frac{\phi_2}{\phi_0} = O(1).$$

Although ϕ_1, ϕ_2 will have solutions of exactly the same form as ϕ_0, the coefficients for these higher-order terms have not been pursued. For the present purposes this is not necessary since ϕ_0 has already demonstrated a dependence on the outer equation.

The above solutions are essentially what has been postulated in the past, but the exact conditions for uniform validity have not been stated before. And the solutions have never presented themselves in such a straightforward (albeit algebraically lengthy) analysis. As mentioned earlier, previous solutions were obtained somewhat heuristically by researchers relying on intuition and skill. The method presented here is straightforward enough to be readily understood using simple second-order differential equations as examples, yet it is still powerful enough to discover results for as difficult a problem as the OS equation, which has been studied since the turn of the century.

Finally, one should note that η is exactly as found via the WKB method [5]. This illustrates the usefulness and validity of the multiple-scales technique when used in conjunction with Kruskal's minimum-simplification technique. The proper form of the stretched variable is determined as a natural result of the method.

3. Velocity profiles with inflection and critical points

A more complicated example than the previous one occurs if one assumes $U - C$ and U'' have simple zeros (at the same point) in the known velocity profile, U. Other types of profiles are possible, but this one is presented because of its relevance to previous work.

More general profiles, where $U - C = 0$ and $U'' = 0$ occur at different points, are not included here but are the subject of current research. For results relevant to these cases one could refer to [1], where they are examined using techniques less straightforward than the multiple-scales method.

The analysis for this case will differ from the one previously discussed beginning with the determination of the exponent ν. Since coefficients other than ε in the OS equation will be getting small (in fact, zero) we must use some technique to account for their smallness in the simplification process. One way to do this automatically is to write them as

$$(U - c) = \frac{\varepsilon^{\nu}\eta}{g}(U - c) \quad \text{and} \quad U'' = \frac{\varepsilon^{\nu}\eta}{g}U''$$

since $\varepsilon^{\nu}\eta = g$ from (2). This allows us to use only the ε-terms in our estimates of the size of each term in the equation; and we are still able to account for the fact that $U - C$ and U'' are both small.

In this case the expanded OS equation takes the form:

$$\begin{aligned}
&\varepsilon^{1-4\nu}g'^4\phi_{\eta\eta\eta\eta} - \varepsilon^{-\nu}\frac{g'^2}{g}(U - c)\eta\phi_{\eta\eta} + \varepsilon^{1-3\nu}g'^2[4g'\phi_{\eta\eta\eta z} + 6g''\phi_{\eta\eta\eta}] \\
&\quad - \frac{g''(U - c)}{g}\eta\phi_{\eta} - \frac{2g'(U - c)}{g}\eta\phi_{\eta z} - \varepsilon^{\nu}\frac{\eta}{g}[(U - c)(\phi_{zz} - \alpha^2\phi) - U''\phi] \\
&\quad + \varepsilon^{1-2\nu}[6g'^2\phi_{\eta\eta zz} + 12g'g''\phi_{\eta\eta z} + (4g'g'' + 3g''^2 - 2\alpha^2 g'^2)\phi_{\eta\eta}] \\
&\quad + \varepsilon^{1-\nu}[4g'\phi_{\eta zzz} + 6g''\phi_{\eta zz} + 4(g''' - \alpha^2 g')\phi_{\eta z} + (g^{iv} - 2\alpha^2 g'')\phi_{\eta}] \\
&\quad + \varepsilon[\phi_{zzzz} - 2\alpha^2\phi_{zz} + \alpha^4\phi] = 0.
\end{aligned} \tag{13}$$

Requiring that the two largest terms be of the same order of magnitude means that $\nu = 1/3$, which is consistent with previous results [3]. Inserting this value into (13) and expanding ϕ in an asymptotic series in powers of $\varepsilon^{1/3}$,

$$\phi(z, \eta) = \phi_0(z, \eta) + \varepsilon^{1/3}\phi_1(z, \eta) + \varepsilon^{2/3}\phi_2(z, \eta) + \cdots$$

yields a different set of governing equations for ϕ_0, ϕ_1, ϕ_2 . . . These are:

$$\phi_{0_{\eta\eta\eta\eta}} - K^3\eta\,\phi_{0_{\eta\eta}} = 0, \tag{14}$$

$$\phi_{1_{\eta\eta\eta\eta}} - K^3\eta\,\phi_{1_{\eta\eta}} = \frac{1}{g'}\left[-4\phi_{0_{\eta\eta\eta z}} - \frac{6g''}{g'}\phi_{0_{\eta\eta\eta}} + \eta K^3\left(2\phi_{0_{\eta z}} + \frac{g''}{g'}\phi_{0_{\eta}}\right)\right], \tag{15}$$

$$\begin{aligned}
\phi_{2_{\eta\eta\eta\eta}} - K^3\eta\phi_{2_{\eta\eta}} &= \frac{1}{g'}\left[-4\phi_{1_{\eta\eta\eta z}} - \frac{6g''}{g'}\phi_{1_{\eta\eta\eta}} + \eta K^3\left(2\phi_{1_{\eta z}} + \frac{g''}{g'}\phi_{1_{\eta}}\right)\right] \\
&\quad - \frac{1}{g'^2}\left[6\phi_{0_{\eta\eta zz}} + \frac{12g''}{g'}\phi_{0_{\eta\eta z}} + \left(\frac{4g'''}{g'} + \frac{3g''^2}{g'^2} - 2\alpha^2\right)\phi_{0_{\eta\eta}}\right] \\
&\quad + \frac{\eta}{gg'^4}[(U - c)(\phi_{0_{zz}} - \alpha^2\phi_0) - U''\phi_0],
\end{aligned} \tag{16}$$

where

$$K(z)^3 = \frac{U - c}{gg'^2}.$$

The solution to (14) is found by defining

$$\psi_0 = \phi_{0_{\eta\eta}}$$

and integrating twice to obtain

$$\phi_0(z, \eta) = A_0(z) \int^\eta \int^\zeta \mathrm{Ai}(K\xi)\,\mathrm{d}\xi\mathrm{d}\zeta + B_0(z) \int^\eta \int^\zeta \mathrm{Bi}(K\xi)\,\mathrm{d}\xi\mathrm{d}\zeta + D_0(z)\eta + E_0(z),$$

where Ai and Bi represent Airy functions [14]. This could also be written in terms of Hankel functions of the third order as done by Heisenberg.

In order to insure the uniform validity of this, all solutions to the homogeneous form of (15) must be eliminated from the right-hand side of (15). This is done, as before, by judiciously choosing the arbitrary functions K (which determines $g(z)$), A_0, B_0, and D_0. It is easily verified that the following satisfy the above criteria:

$$K' = 0 \quad \text{or} \quad K = 1,$$

$$\frac{A_0'}{A_0} + \frac{5}{2}\frac{g''}{g'} = 0, \quad \frac{B_0'}{B_0} + \frac{5}{2}\frac{g''}{g'} = 0, \quad \frac{D_0'}{D_0} + \frac{1}{2}\frac{g''}{g'} = 0.$$

Note that these are identical to those in the previous problem, but in this case they result in:

$$g(z) = \left[\frac{3}{2}\int^z \sqrt{U - c}\,\mathrm{d}z\right]^{2/3},$$

$$A(z) = a(U - c)^{-5/4}\left[\frac{3}{2}\int^z \sqrt{U - c}\,\mathrm{d}z\right]^{5/6},$$

$$B_0(z) = b(U - c)^{-5/4}\left[\frac{3}{2}\int^z \sqrt{U - c}\,\mathrm{d}z\right]^{5/6},$$

$$D_0(z) = d(U - c)^{-1/4}\left[\frac{3}{2}\int^z \sqrt{U - c}\,\mathrm{d}z\right]^{1/6},$$

where a, b, and d are again arbitrary constants. The expression for g(z) results from choosing $K = 1$ since $K^3 = (U - c)/gg'^2$. It is important to note that in the above analysis $g(z)$ is a natural byproduct and results in the same stretched variable (or Langer variable) as shown appropriate by Tollmien [4, 6]. No previous knowledge of this function was necessary. Also, the transformation represented by $g(z)$ was used only on the stretched variable. Tam [7] used it to transform the entire Orr-Sommerfeld equation resulting in extremely complicated nonlinear terms.

Just as in the previous example, up to this point nothing has been said of $E_0(z)$. Looking at the forcing function for ϕ_2 one sees that, as before, it must satisfy the Rayleigh equation in order to insure uniform validity of the solution. Furthermore, $D_0(z)$ must be identically zero also. Therefore the uniformly valid solution to the OS equation assuming $U - C = 0$ and $U'' = 0$ at the same point, is

$$\phi_0(z, \eta) = \frac{\left[\frac{3}{2}\int\sqrt{U-c}\,\mathrm{d}z\right]^{5/6}}{(U-c)^{5/4}}\left[a\int^{\eta}\int^{\zeta}\mathrm{Ai}(\zeta)\,\mathrm{d}\xi\mathrm{d}\zeta + b\int^{\eta}\int^{\zeta}\mathrm{Bi}(\xi)\,\mathrm{d}\xi\mathrm{d}\zeta\right]$$

$$+ E_{0_1}(z) + E_{0_2}(z) + \mathrm{O}(|\varepsilon|^{2/3})$$

where

$$(U - c)(E''_{0_{1,2}}(z) - \alpha^2 E_{0_{1,2}}(z)) - U''E_{0_{1,2}}(z) = 0.$$

This is uniformly valid as $\varepsilon \to 0$ for all z such that $g(z) = \mathrm{O}(1)$.

The above solution reproduces Tollmien's improved viscous formulae or solutions to the Rayleigh equation depending on whether $z = \mathrm{O}(1)$ or $\eta = \mathrm{O}(1)$. They are both included in a single expression without recourse to asymptotic matching or the assumption of an overlap region. In addition, for the special case of a linear velocity profile the above solution reduces to that of Tam's equation 2.21.

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